Emergent Electrodynamics in Helium-3

Jarramplas'13

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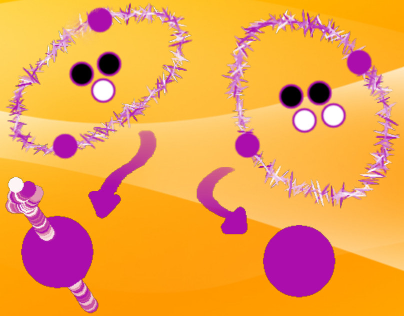
Summary

^{4}He and ^{3}He

The simplest quantum liquids are the liquid phases of the two stable isotopes of helium ^{4}He and ^{3}He.

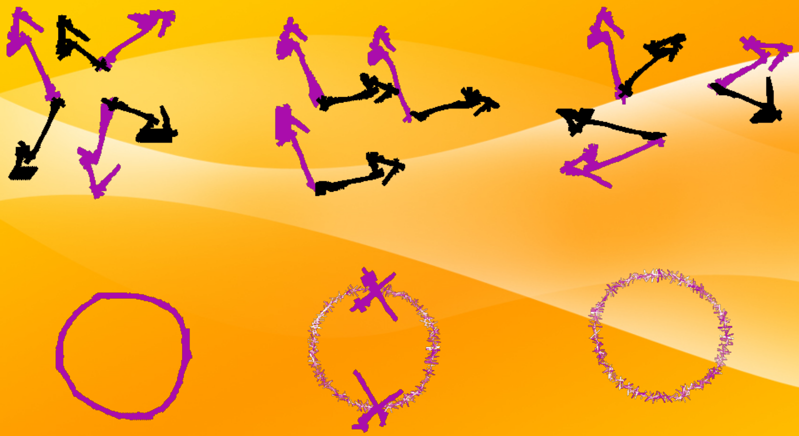


At low temperatures these atoms are in the electronic ground state. Two helium atoms interact via a Lennard-Jones potential. In addition, for ^{3}He there is an interaction between the nuclear magnetic dipole moments.

Helium-3: Phase diagram

- What the experiments say: Two superfluid phases.
- What the microscopic theory says: Broken symmetries and topology of momentum space.
    - Normal phase: Fermi surface.
    - A-phase: Fermi points.
    - B-phase: Fully gapped.

Helium-3: Symmetries and momentum space



Helium-3: Fundamentals

- Bose-Einstein condensation: Macroscopic ocupation of a quantum state in a system of identical bosons.
- Cooper's problem: Consider two electrons subject to an attractive interaction. If we put these electrons over the zero-temperature vacuum state of $N-2$ electrons, then there always exists a bound state (overall energy E below twice the Fermi level) no matter how small the interaction strength is.

Helium-3: Fundamentals

- Angular momentum: Cooper pairs form with an overall angular momentum l when V_l is the most attractive component (most negative) of the interaction potential. In particular, $l \neq 0$ is an option.
- Spin-triplet p-wave pairing: Cooper pairs are formed from states near the Fermi surface. They have a relative momentum of order $2p_{\mathsf{F}}$. Moreover, the hard-core repulsion prevents two ^{3}He atoms from getting closer to one another than a distance $r_0 \sim 2.5\mathring{A}$. Cooper pairs in ^{3}He must have then a nonzero angular momentum, in contraposition to the usual BCS theory in superconductors.

Ginzburg-Landau theory

- Condensed phase: Ginzburg-Landau theory of second-order phase transitions with order parameter

$$e^i_a. \tag{1}$$

Transforms under the $S = 1$, $L = 1$ representations of $\mathrm{SO}(3)_S$ (a index) and $\mathrm{SO}(3)L$ (i index): realization of spin-triplet p-wave pairing.

- Phenomenological approach near the critical temperature: below T_c the free energy difference between the ordered and normal phases can be expanded as

$$\Delta F := F - F_n = \alpha_0(T - T_\mathrm{c})I_0 + \frac{1}{2}\beta(T_\mathrm{c})\sum_{s=1}^{5}\beta_s J_s, \tag{2}$$

as long as we are near the critical temperature and the free energy functional is continuous.

Invariants

The six invariants are given by:

$$I_0 := \sum_{i,a} e^i_a e^{i*}_{\ a}, \tag{3}$$

$$J_1 := \sum_{i,a} \sum_{j,b} e^i_a e^i_a e^{j*}_{\ b} e^{j*}_{\ b}, \tag{4}$$

$$J_2 := \sum_{i,a} \sum_{j,b} e^i_a e^j_b e^{i*}_{\ a} e^{j*}_{\ b}, \tag{5}$$

$$J_3 := \sum_{i,a} \sum_{j,b} e^i_a e^j_a e^{i*}_{\ b} e^{j*}_{\ b}, \tag{6}$$

$$J_4 := \sum_{i,a} \sum_{j,b} e^i_a e^j_b e^{j*}_{\ a} e^{i*}_{\ b}, \tag{7}$$

$$J_5 := \sum_{i,a} \sum_{j,b} e^i_a e^i_b e^{j*}_{\ a} e^{j*}_{\ b}. \tag{8}$$

Unitarity condition

- No way of solving the minimization problem. Unitarity condition:
$$\sum_{b,c} \epsilon_{abc} e^{i*}_{\ b} e^{j}_{c} = 0. \tag{9}$$
- All the evidence is that the states of ^{3}He which are realized in Nature are unitary.
- Within the space of unitary states, four solutions can be found.

Unitary states

- BW state (Balian and Werthamer, 1963). Reduced spin susceptibility from the normal-state value: Natural identification with the experimentally observed B-phase.

$$e_a^i = \Delta_0 \delta_a^i. \tag{10}$$

- ABM state (Anderson and Morel, 1961; Anderson and Brinkman, 1973). Nuclear magnetic resonance evidence points to this state as a succesful description of the A-phase.

$$e_a^i = \Delta_0 \hat{z}_a (\hat{m}^i + i\hat{n}^i). \tag{11}$$

Unitary states

- Planar state: Local minimum.

$$e^i_a = \Delta_0(\hat{d}'_a \hat{m}^i + \hat{d}''_a \hat{n}^i). \tag{12}$$

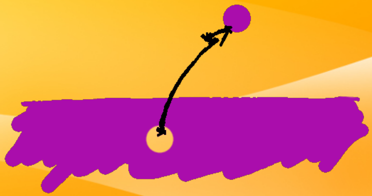
Energetically unstable toward the fully gapped B-phase.

- Polar state: Nodal lines are marginal.

$$e^i_a = \Delta_0 \hat{d}_a \hat{m}^i. \tag{13}$$

Landau Fermi-liquid theory

- Excitations of a Fermi gas: take a fermion below the Fermi level and promote it to a fermion with $k > k_{\mathsf{F}}$.



- Hypothesis to treat the system when the interaction between particles is turned on gradually: The ground state of the noninteracting system evolves adiabatically into the ground state of the interacting system, and similarly each excited state of the noninteracting one into a corresponding excited state of the interacting one.

Landau Fermi-liquid theory

- The Hamiltonian for the quasiparticles is given by

$$\hat{H} := \hat{T} + \hat{V} = \sum_{\boldsymbol{p}\sigma} \epsilon(p) a^{+}_{\boldsymbol{p}\sigma} a_{\boldsymbol{p}\sigma} + \sum_{ijkl} V_{ijkl} a^{+}_{i} a^{+}_{j} a_k a_l. \quad (14)$$

- The quasiparticle kinetic energy is given by

$$\epsilon(p) = \epsilon_{\mathsf{F}} + \frac{p_{\mathsf{F}}}{m^*}(p - p_{\mathsf{F}}). \quad (15)$$

Landau Fermi-liquid theory

- The interaction term will be taken to be generic within the constraints imposed by symmetry. The spin dependence splits in the singlet and triplet states. Since we want to study $l = 1$ pairing, it is enough to consider a potential with only $V_1 \neq 0$, which is proportional to the scalar product
$$\boldsymbol{p} \cdot \boldsymbol{p}'. \tag{16}$$
Moreover, the most important interaction terms to the formation of Cooper pairs, the so-called pairing terms, are those corresponding to the scattering of two quasiparticles with CoM momentum zero:
$$\hat{V} = -\sum_{\alpha\beta} \left(\sum_{\boldsymbol{p}'} \boldsymbol{p}' a^{\dagger}_{\boldsymbol{p}'\alpha} a^{\dagger}_{-\boldsymbol{p}'\beta} \right) \cdot \left(\sum_{\boldsymbol{p}} \boldsymbol{p} a_{\boldsymbol{p}\alpha} a_{-\boldsymbol{p}\beta} \right). \tag{17}$$

- Ground-state wavefunction:

$$\psi_0 := \mathcal{N} \left(\sum_{\boldsymbol{k}} c_{\boldsymbol{k}\alpha\beta} a^+_{\boldsymbol{k}\alpha} a^+_{-\boldsymbol{k}\beta} \right) |0\rangle. \tag{18}$$

Some quantities acquire nonzero expectation values:

$$\boldsymbol{\Psi}^i_{\alpha\beta} := \left\langle \sum_{\boldsymbol{p}} p^i a_{\boldsymbol{p}\alpha} a_{-\boldsymbol{p}\beta} \right\rangle . \tag{19}$$

This quantity is nothing but the order parameter we used in the Ginzburg-Landau phenomenological approach in disguise: it can be always written as

$$\boldsymbol{\Psi}^i_{\alpha\beta} := i\boldsymbol{e}^i_a (\sigma^a \sigma^2)_{\alpha\beta}. \tag{20}$$

^{3}He Hamiltonian (i)

- Hamiltonian:

$$\mathcal{H} - \mu N = \sum_{\boldsymbol{p},\alpha} M_{\boldsymbol{p}} a^{\dagger}_{\boldsymbol{p}\alpha} a_{\boldsymbol{p}\alpha} -$$

$$- \sum_{\alpha\beta} \left(\sum_{\boldsymbol{p}'} \boldsymbol{p}' a^{\dagger}_{\boldsymbol{p}'\alpha} a^{\dagger}_{-\boldsymbol{p}'\beta} \right) \cdot \left(\sum_{\boldsymbol{p}} \boldsymbol{p} a_{\boldsymbol{p}\alpha} a_{-\boldsymbol{p}\beta} \right)$$

with $\quad M_{\boldsymbol{p}} = \boldsymbol{p}^2/2m - \mu.$

^{3}He Hamiltonian (ii)

- The order parameter characterises the state of most of the atoms, which are paired with vanishing total momentum:

$$\Psi^i_{\alpha\beta} = \left\langle \sum_{\boldsymbol{p}} p^i a_{\boldsymbol{p}\alpha} a_{-\boldsymbol{p}\beta} \right\rangle = i e^i_a \sigma^a \sigma^2,$$

  - σ^a, $a = 1, 2, 3$ are Pauli matrices.
  - e^i_a is then a vector under
    - spatial rotations (index i)
    - the spin rotation group (index a).

^{3}He Hamiltonian (iii)

- Linearise around this vacuum expectation value:

$$\mathcal{H} - \mu N = \boldsymbol{\Psi}^{\dagger}_{\alpha\beta} \cdot \boldsymbol{\Psi}^{\alpha\beta} + \sum_{\boldsymbol{p}} \mathcal{H}_{\boldsymbol{p}}$$

$$\mathcal{H}_{\boldsymbol{p}} = M_{\boldsymbol{p}} \sum_{\alpha} a^{\dagger}_{\boldsymbol{p}\alpha} a_{\boldsymbol{p}\alpha} + \frac{1}{2}\left[\boldsymbol{\Psi}^{\dagger\alpha\beta} \cdot \left(\boldsymbol{p} a_{-\boldsymbol{p}\alpha} a_{\boldsymbol{p}\beta}\right) + \text{H.c.}\right]$$

- Equations of motion for $a_{\boldsymbol{p}\alpha}$:

$$i\dot{a}_{\boldsymbol{p}\alpha} = M_{\boldsymbol{p}} a_{\boldsymbol{p}\alpha} + \boldsymbol{p} \cdot \boldsymbol{\Psi}_{\alpha}{}^{\beta} a^{\dagger}_{-\boldsymbol{p}\beta}.$$

These equations no longer exhibit the U(1) invariance of the original Hamiltonian

Planar phase (i)

- The order parameter for this phase is

$$\boldsymbol{e}_a = \delta_{a1}\hat{\boldsymbol{m}} + \delta_{a2}\hat{\boldsymbol{n}} \quad \Rightarrow \quad \boldsymbol{\Psi} = -\hat{\boldsymbol{m}}\sigma^3 + i\hat{\boldsymbol{n}}\mathbb{1}$$

- The Hamiltonian becomes (up to additive constants):
  - $\sum_{\boldsymbol{p}} \mathcal{H}_{\boldsymbol{p}} = \sum_{\boldsymbol{p}} \chi^\dagger_{\boldsymbol{p}} H_{\boldsymbol{p}} \chi_{\boldsymbol{p}}, \qquad \Delta_{\boldsymbol{p}} = -(p_m - ip_n).$
  - $\chi_{\boldsymbol{p}} = \begin{pmatrix} a_{\boldsymbol{p}\uparrow} \\ a^\dagger_{-\boldsymbol{p}\downarrow} \end{pmatrix}, \qquad H_{\boldsymbol{p}} = \begin{pmatrix} M_{\boldsymbol{p}} & \Delta_{\boldsymbol{p}} \\ \Delta^*_{\boldsymbol{p}} & -M_{\boldsymbol{p}} \end{pmatrix}.$
- Equations of motion: $\quad i\partial_t \chi_{\boldsymbol{p}} = H_{\boldsymbol{p}} \chi_{\boldsymbol{p}}$
  - U(1) invariant.
  - Vanishing Noether charge.

Planar phase (ii)

- Eigenvalues of $H_{\boldsymbol{p}}$: $\quad \pm\varepsilon_{\boldsymbol{p}}^2, \qquad \varepsilon_{\boldsymbol{p}} = \sqrt{M_{\boldsymbol{p}}^2 + |\Delta_{\boldsymbol{p}}|^2}.$

- $\varepsilon_{\boldsymbol{p}}$ vanishes when $\Delta_{\boldsymbol{p}} = M_{\boldsymbol{p}} = 0 \quad \Rightarrow$

  Fermi points: $\quad \boldsymbol{p} = q p_{\mathrm{F}} \hat{\boldsymbol{l}}, \quad q = \pm 1, \quad p_{\mathrm{F}} = \sqrt{2m\mu}.$

- Excitations.

  A little energy will produce a bunch of quasiparticles with momentum near the Fermi points and, moreover, with relativistic dispersion relation,

$$\boldsymbol{p} = q p_{\mathrm{F}} \hat{\boldsymbol{l}} + \mathbf{p}, \qquad \varepsilon_{\mathbf{p}}^q = |\mathbf{p}|, \qquad |\mathbf{p}| \ll p_{\mathrm{F}}.$$

Planar phase (iii)

- An internal observer, who is made of and only sees the excitations described above, can describe this low-energy regime using scattering experiments:
  - Incoming particles (1,2) from the *same* Fermi point $\rightarrow$ outgoing particles (3,4) from the *same* Fermi point
    - Momentum conservation:

$$(qp_{\mathrm{F}}\hat{\boldsymbol{l}} + \mathbf{p}_1) + (qp_{\mathrm{F}}\hat{\boldsymbol{l}} + \mathbf{p}_2) = \boldsymbol{p}_3 + \boldsymbol{p}_4 \quad \Rightarrow$$
$$\Rightarrow \quad \boldsymbol{p}_3 = qp_{\mathrm{F}}\hat{\boldsymbol{l}} + \mathbf{p}_3, \qquad \boldsymbol{p}_4 = qp_{\mathrm{F}}\hat{\boldsymbol{l}} + \mathbf{p}_4.$$

    - Energy conservation: $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$.

Planar phase (iv)

- Incoming particles (1,2) from the *different* Fermi points $\rightarrow$ outgoing particles (3,4) from the *different* Fermi points
  - Momentum conservation:

$$(+p_{\rm F}\hat{\boldsymbol{l}} + \mathbf{p}_1) + (-p_{\rm F}\hat{\boldsymbol{l}} + \mathbf{p}_2) = \boldsymbol{p}_3 + \boldsymbol{p}_4 \quad \Rightarrow$$
$$\Rightarrow \quad \boldsymbol{p}_3 = q p_{\rm F}\hat{\boldsymbol{l}} + \mathbf{p}_3, \qquad \boldsymbol{p}_4 = -q p_{\rm F}\hat{\boldsymbol{l}} + \mathbf{p}_4.$$

  - Energy conservation: $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$.

Planar phase (v)

- Up to now we have assumed a homogeneity.
- If we allow for smooth changes in density, at each space point, the Fermi points will be of the form $q[p_{\mathrm{F}} + \delta A(\boldsymbol{x})]\hat{\boldsymbol{l}}(\boldsymbol{x})$ and $\boldsymbol{m}(\boldsymbol{x})$, $\boldsymbol{n}(\boldsymbol{x})$
- Independent degrees of freedom:

$$a_{\boldsymbol{p}\uparrow} \quad \rightarrow \quad \alpha_{\mathbf{p}\uparrow} := a_{\boldsymbol{p}_{\mathrm{F}}+\mathbf{p}\uparrow}, \quad \beta_{\mathbf{p}\uparrow} := a_{-\boldsymbol{p}_{\mathrm{F}}+\mathbf{p}\uparrow}$$

- Evolution equation for $\chi_{\mathbf{p}\uparrow} = (\alpha_{\mathbf{p}\uparrow}, \beta^{\dagger}_{-\mathbf{p}\uparrow})^{\mathrm{T}}$

$$i\partial_t \chi_{\mathbf{p}\uparrow} = H_{\mathbf{p}\uparrow}\chi_{\mathbf{p}\uparrow},$$
$$H_{\mathbf{p}\uparrow} = (p_l - \delta A)\sigma^3 - p_m\sigma^1 - p_n\sigma^2$$

Planar phase (vi)

- It might seem that we have doubled the degrees of freedom in a Fermi point, but the other Fermi point is already described by this equation:

$$\chi_{\mathbf{p}\uparrow} \to \chi'_{\mathbf{p}\uparrow} = \sigma^1 \chi^*_{\mathbf{p}\uparrow} \quad \text{gives equation with } q = -1$$

- The same applies to $\downarrow$ projection
- Go to to position representation $\quad \mathbf{p} \to -i\boldsymbol{\partial}$
- $\chi_{\uparrow\downarrow}$ evolve according to $\quad i\partial_t \chi_{\uparrow\downarrow} = H_{\uparrow\downarrow}\chi_{\uparrow\downarrow}$,

$$H_{\uparrow\downarrow} = \sigma^3(-i\partial_l - \delta A) \pm i\sigma^1\partial_m + i\sigma^2\partial_n$$

- Helicity is given by $\quad h_{\uparrow\downarrow} = (+1)\cdot(\mp 1)\cdot(-1) = \pm 1$

Planar phase (vii)

Finally, the bispinor $\psi = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix}$ obeys the Dirac equation $\gamma^\mu(-i\partial_\mu - A_\mu)\psi = 0$ where

$$A_0 = A_n = A_m = 0, A_l = \delta A$$

and

$$\gamma^0 = \left(\begin{array}{cc} 0 & \sigma_1 \\ \sigma_1 & 0 \end{array} \right), \qquad \gamma^1 = \left(\begin{array}{cc} 0 & I_2 \\ -I_2 & 0 \end{array} \right),$$
$$\gamma^2 = \left(\begin{array}{cc} 0 & -i\sigma_3 \\ -i\sigma_3 & 0 \end{array} \right), \qquad \gamma^3 = \left(\begin{array}{cc} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{array} \right).$$

Planar phase (viii)

Helicity/chirality. Construct $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and the chirality projectors $C_\pm = (\mathbb{1} \pm \gamma^5)/2$ Then

$$C_+\psi = \chi_\uparrow, \qquad C_-\psi = \chi_\downarrow$$

Therefore, chirality is an emergent low-energy concept

- incarnated by the atomic spin $\uparrow\downarrow$
- tied up to the duality of the Fermi points

Planar phase (ix)

Charge is also emergent

- the low-energy evolution (Dirac) equation is U(1) invariant
- the corresponding Noether charge $Q = \int :\psi^\dagger \psi:$ is conserved.

$$Q = Q_\uparrow + Q_\downarrow = \sum_{\mathbf{p}\alpha} (\alpha^\dagger_{\mathbf{p}\alpha} \alpha_{\mathbf{p}\alpha} - \beta^\dagger_{\mathbf{p}\alpha} \beta_{\mathbf{p}\alpha})$$

A-phase

- The order parameter in this phase is

$$\boldsymbol{e}_a = (\hat{\boldsymbol{m}} + i\hat{\boldsymbol{n}})\delta_{a3} \quad \Rightarrow \quad \boldsymbol{\Psi} = i(\hat{\boldsymbol{m}} + i\hat{\boldsymbol{n}})\sigma^1$$

- In order to compare with the planar phase, we define

$$a_{\boldsymbol{p}\rightleftarrows} = (a_{\boldsymbol{p}\uparrow} \pm a_{\boldsymbol{p}\downarrow})/\sqrt{2}$$

- In the position representation, $\chi_{\rightleftarrows}$ evolve according to

$$H_{\rightleftarrows} = \sigma^3(-i\partial_l - \delta A) \mp i\sigma^1\partial_m \pm i\sigma^2\partial_n$$

- Helicity is given by $h_{\rightleftarrows} = (+1)\cdot(\pm 1)\cdot(\mp 1) = -1.$
  The same for both components $\rightleftarrows$.
  So excitations, represented by these spinors, cannot be described by a Dirac bispinor

Thank you for your attention.